STATIONARY DUST-FILLED COSMOLOGICAL SOLUTION WITH $\Lambda = 0$ AND WITHOUT CLOSED TIME-LIKE LINES

by

S.C. Maitra

TECHNICAL REPORT NO. 533

December 1965

[redacted]

UNIVERSITY OF MARYLAND
DEPARTMENT OF PHYSICS AND ASTRONOMY
COLLEGE PARK, MARYLAND

STATIONARY DUST-FILLED COSMOLOGICAL SOLUTION WITH $\Lambda = 0$ AND WITH-OUT CLOSED TIME-LIKE LINES

by

S.C. Maitra
University of Maryland
Department of Physics and Astronomy
College Park, Maryland

ABSTRACT

An analytic and complete solution of Einstein's field equations without the Λ-term is presented for a dust-filled universe ($p = 0$). The solution is stationary and inhomogeneous and does not contain any closed time-like lines. Also some of the properties of the solution are studied.

[redacted]

INTRODUCTION

The purpose of this paper is to present an analytic and complete solution of Einstein's field-equations for a dust-filled universe ($\rho > 0$, $p = 0$) without the cosmological Λ-term. The solution is inhomogeneous and stationary, with cylindrical symmetry, so it will not be found appropriate in discussions of observational cosmology; but its existence may give reason to hope that there may also exist non-stationary solutions which avoid the singular epochs found in the Friedman solutions and other related cosmological models. The solution presented here has the further merit that it does not contain any closed time-like lines. All known solutions of Einstein's field equations for a dust-filled universe seem to suffer from some undesirable features. Consider first the stationary solution. The spatially homogeneous solutions of Einstein, Gödel[1], Ozsvath and Schücking[2], and Ozsvath[3] all require a non-vanishing cosmological Λ-term. The inhomogeneous solution of Lanczos[4] and van Stockum[5] (which has recently been re-discovered by Wright[6]) although it does

---

(1) Gödel, K., Rev. Mod. Phys. 21, 447 (1949).

(2) Ozsvath, I. and Schücking, E., Nature, 193, 1168 (1962).

(3) Ozsvath, I., J. Math. Phys. 6, 591 (1965).

(4) Lanczos, Zeits. f. Physik, 21, 73 (1924).

(5) van Stockum, W.J., Proc. Roy. Soc. Ed. 57, 135 (1937).

(6) Wright, J.P., J. Math. Phys. 6, 103 (1965).

not require a cosmological term, nevertheless contains closed time-like lines as in Gödel's universe. Further, as pointed out by Shepley[7], the solution has a singularity at a finite proper distance from the axis of symmetry where the matter-density and scalar curvature become infinite. Ehlers[8] has shown how one can construct all solutions with $\Lambda = 0$ for distributions of matter in rigid rotation from static vacuum metrics, but the global properties regarding the presence of singularities and closed time-like lines have not been investigated. Of the non-stationary solutions with vanishing cosmological constant, the singular epochs in the Friedman homogeneous and isotropic solutions are familiar, and Shepley[7] has shown that a large class of closed, homogeneous, non-isotropic solutions also involve singular epochs, while Hawking[9] shows that all solutions which at some epoch differ from the open Friedman model in sufficiently small but otherwise arbitrary ways have, like the Friedman model itself, evolved from a singular beginning. We leave as a problem for further investigations to decide whether the present example is entirely exceptional, or whether it is an especially simple limiting case for some significant class of non-singular cosmologies.

---

(7) Shepley, L., Proc. Nat. Acad. Scie. U.S.A. 52, 1403 (1964).

(8) Ehlers, J., Recent Developments in General Relativity (Pergamon Press, New York, 1962, p. 201).

(9) Hawking, S.W., preprint. See also Hawking, S. and Ellis, G.F.R., Phys. Letters 17, 246 (1965).

I. Statement of Results

The solution given in eqs. I.1 - I.5 below has the following properties.

1) It is cylindrically symmetric.

2) It is stationary but the time-like Killing vector is not the velocity-vector of matter.

3) Defined with respect to velocity vector of matter, shear and rotation do not vanish but the expansion vanishes. Thus unlike the case of Lanczos-van Stockum and Ehlers solutions the motion is non-rigid.

4) It does not contain any closed time-like line.

5) The space is complete.

6) The solution is open in all spatial directions, i.e. it extends to infinite proper distance in all directions.

7) Matter everywhere moves in circles about the axis of symmetry.

8) The solution is spatially inhomogeneous and the density as

well as the kinematic quantities rotation and shear tend to zero as one goes to arbitrarily large distances from the axis of symmetry.

Proof of the non-obvious statements will be given later.

The line element

$$-ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = -dt^2 + e^{2\Psi}(dr^2 + dz^2) + (r^2 - m^2)d\varphi^2$$

$$+ 2m d\varphi dt \qquad \text{I(1)}$$

is a solution of the Einstein field equations

$$R_{\mu\nu} - \tfrac{1}{2}R g_{\mu\nu} = -KT_{\mu\nu} \qquad \text{I(2)}$$

with

$$T_{\mu\nu} = \rho v_{\mu} v_{\nu} \qquad \text{I(3)}$$

where v^{μ} is a unit time-like vector and constitutes a geodesic congruence and ρ is the matter density with following values for Ψ and m:

$$\Psi = [- \frac{1}{4x^2} \{(1 + x^2)^{\frac{1}{2}} - 1\} + \frac{1}{8} - \tfrac{1}{4}\ln \frac{(1 + x^2)^{\frac{1}{2}} + 1}{2}] \qquad \text{I(4)}$$

$$m = \frac{a}{2}\left[(1 + x^2)^{\frac{1}{2}} - 1 - \ln \frac{(1 + x^2)^{\frac{1}{2}} + 1}{2}\right] \qquad \text{I(5)}$$

Here we have introduced a new variable x through

$$r = \frac{a}{2} \cdot x$$

where a is a constant.

It is to be noted that by introducing the co-ordinates

$$\begin{aligned} \chi &= \frac{2}{a}r \\ \tau &= \frac{2}{a}t \\ \xi &= \frac{2}{a}z \\ \varphi &= \varphi \end{aligned}$$

we can write

$$ds^2 = \frac{a^2}{4}(ds^o)^2$$

where $(ds^o)^2$ depends only $\chi\tau\xi\varphi$ and does not contain a. Hence this constant can actually be reduced to a scale factor by a co-ordinate transformation and does not play any further role.

The non-vanishing components of v^μ are

$$v^\varphi = \frac{-y}{r(1 - y^2)^{\frac{1}{2}}} \qquad \text{I(6)}$$

$$v^t = \frac{r - my}{r(1 - y^2)^{\frac{1}{2}}} \tag{I(7)}$$

where

$$y = m' = \frac{dm}{dr} \tag{I(8)}$$

The density is given by

$$K\rho = \frac{4}{a^2} e^{-2\Psi} \cdot \frac{1}{x^4(1 + x^2)^{\frac{1}{2}}} \{(1 + x^2)^{\frac{1}{2}} - 1\}^2 \tag{I(9)}$$

II. Computed Properties

A number of properties of the solution stated above follows from straightforward computations, the results of which will be given here.

It can easily be shown from the expression I(9) of density that the total amount of matter (calculated per unit proper length along z-direction) is finite whereas the total proper volume (per unit proper length in z-direction) is infinite so that the matter is distributed with zero average density or in infinite dilution.

The vorticity vector corresponding to the above velocity is defined by

$$\omega^{\mu} = \frac{1}{2(-g)^{\frac{1}{2}}} \epsilon^{\mu\alpha\beta\gamma} v_{\alpha} \frac{\partial v_{\beta}}{\partial x^{\gamma}} \qquad \text{II(1)}$$

$$\epsilon^{0123} = +1$$

It has only a component in z-direction, and the magnitude of angular velocity is given by

$$\omega^2 = g_{zz}\omega^z\omega^z = \tfrac{1}{4} \cdot \frac{1}{a^2} e^{-2\Psi} \cdot \frac{1}{1 + x^2} \qquad \text{II(2)}$$

The shear tensor

$$\varphi_{\mu\nu} = \tfrac{1}{2}(v_{\mu;\nu} + v_{\nu;\mu}) - \tfrac{1}{3}(g_{\mu\nu} + v_{\mu}v_{\nu})v^{\rho}{}_{;\rho} \qquad \text{II(3)}$$

has only non-vanishing components φ^{tr} and $\varphi^{\varphi r}$ given by

$$\varphi^{r\varphi} = \frac{1}{a^2} \left(\frac{x}{2y}\right)^{\frac{1}{2}} e^{-2\Psi} \frac{\{(1 + x^2)^{\frac{1}{2}} - 1\}^2}{x^3(1 + x^2)^{\frac{1}{2}}} \qquad \text{II(4)}$$

$$\varphi^{tr} = -\frac{a}{2} \ln \frac{(1 + x^2)^{\frac{1}{2}} + 1}{2} \varphi^{r\varphi} \qquad \text{II(5)}$$

From this one gets

$$\varphi^2 = \varphi_{\alpha\beta}\varphi^{\alpha\beta} = \frac{1}{2a^2} e^{-2\Psi} \frac{\{(1 + x^2)^{\frac{1}{2}} - 1\}^4}{x^4(1 + x^2)} \qquad \text{II(6)}$$

From the above expression it follows that

$$\tfrac{1}{2}K\rho + \varphi^2 = 2\omega^2 \qquad \text{II(7)}$$

It may be noted that as $r \to \infty$, ρ vanishes more rapidly than φ^2 and ω^2.

We give below the components of Riemann tensor computed in an orthonormal frame. We write

$$ds^2 = -(\omega^o)^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 = g_{\mu\nu}\omega^\mu\omega^\nu$$

where

$$g_{\mu\nu} = \mathrm{diag}(+1, +1, +1, -1) \qquad \text{II(8)}$$

and

$$\begin{aligned}
\omega^1 &= e^{\Psi} . dr \\
\omega^2 &= e^{\Psi} . dz \\
\omega^3 &= r . d\varphi \\
\omega^o &= dt - m d\varphi
\end{aligned} \qquad \text{II(9)}$$

We compute the curvature tensor using the methods described by Misner[10]. Referred to above orthonormal frame, it has the following independent nonvanishing components

$$R_{1212} = -e^{-2\Psi} . \Psi''$$

(10) Misner, C.W., J. Math. Phys., 4, 924 (1963).

$$R_{1310} = \frac{e^{-2\Psi}}{r}\left(\frac{y}{2r} + \tfrac{1}{2}\Psi' y - \tfrac{1}{2}y'\right)$$

$$R_{1010} = e^{-2\Psi}\,\tfrac{1}{4}\,\frac{y^2}{r^2}$$

$$R_{2323} = -e^{-2\Psi}\frac{\Psi'}{r}$$

$$R_{2320} = -e^{-2\Psi}\frac{y}{2r}\,\Psi'$$

$$R_{3030} = \tfrac{1}{4}\,\frac{y^2}{r^2}\,e^{-2\Psi} \qquad \mathrm{II}(10)$$

It will be noted that all components go to zero as $r^{-3/2}$ as $r \to \infty$. The proper radial distance r_p out to a radius r goes as $\int^r e^{\Psi} dr = Kr^{3/4}$ for $r \to \infty$, K being a constant. Hence we see that as $r_p \to \infty$, all components fall off like r_p^{-2}. This is significant in this frame since, with each $g_{\mu\nu} = \pm 1$, one sees that every invariant polynomial in $R_{\mu\nu\alpha\beta}$ will also vanish as $r \to \infty$. In other frames where $g_{\mu\nu}$ depends on r the behaviour of curvature invariants is not easily deduced from that of curvature components.

Finally we compute the c-energy scalar as introduced by Thorne(11) for our system. At a point $(trz\varphi)$ it is defined as

$$U = \frac{1}{8}\left(1 + \frac{A_{,\mu} A^{,\mu}}{4\pi^2 |\vec{\xi}_z|^2}\right) \qquad \mathrm{II}(11)$$

---

(11) Thorne, K.S., Ph.D. Thesis. Unpublished, Princeton University (1965). It should be noted that this definition is slightly different from and is superior to that given in Thorne, K.S., Phys. Rev. 138, B251 (1965).

where $A_{,\mu}$ is the space-time gradiant of the area A of the invariant surface passing through the point and consisting of the points $(t, r, z+\alpha, \varphi+\beta)$ where $0 \le \alpha \le 1$, $0 \le \beta \le 2\pi$ and $|\vec{\xi}_z|$ is the length of the standard "translation" Killing vector at that point. For our metric, we have

$$U = \frac{1}{8}[1 - e^{-2\Psi}\cdot q\{\Psi' + \frac{q'}{2q}\}^2] \qquad \text{II(12)}$$

where we have introduced

$$q = r^2 - m^2 \qquad \text{II(13)}$$

As $r \to \infty$, $U \to \frac{1}{8}$.

III. Analyticity

Using the metric components from Eq. I(1) one computes

$$(-g)^{\frac{1}{2}} = re^{2\Psi} \qquad \text{III(1)}$$

This metric is consequently singular at $r = 0$ where $(-g)^{\frac{1}{2}} = 0$, but is analytic for $r > 0$ where each of the metric components $r^2 - m^2$, m and $e^{2\Psi}$ is analytic and where $(-g)^{\frac{1}{2}} > 0$. We show then that this $r = 0$ singularity is spurious (removable) by interpreting $tr\varphi z$ as cylindrical co-ordinates; that is we introduce new co-ordinates tXYZ by the transformation

$$\begin{aligned} t &= t \\ Z &= z \\ X &= r\cos\varphi \\ Y &= r\sin\varphi \end{aligned} \qquad \text{III(2)}$$

and discuss analyticity in the new co-ordinates. The Jacobian of the transformation is just r, so the metric remains analytic in the region $r^2 = X^2 + Y^2 > 0$ and we need consider in detail only the neighborhood of $r = 0$. To transform Eq. I(1) to these "rectangular" co-ordinates it is most convenient to write

$$\begin{aligned} ds^2 = [-dt^2 + dr^2 + dz^2 + r^2 d\varphi^2] + 2dt(md\varphi) - (md\varphi)^2 \\ + (e^{2\Psi} - 1)(dr^2 + dz^2) \end{aligned} \qquad \text{III(3)}$$

where the quantity in square bracket is, by a familiar computation, analytic (even flat) in tXYZ co-ordinates. We show now that the remaining terms contribute analytic functions to the metric components as is obvious for the term $(e^{2\Psi} - 1)dz^2$ (which contributes $(e^{2\Psi} - 1)$ to g_{ZZ}) since $e^{2\Psi}$ is an analytic function of $r^2 = X^2 + Y^2$, and hence of X,Y. We next note that

$$rdr = XdX + YdY$$

and

$$r^2 d\varphi = -YdX + XdY$$

are analytic differential forms, so the analyticity of the contributions from

$$2dt(md\varphi) = 2\frac{m}{r^2}\, dt(r^2 d\varphi)$$

and $(md\varphi)^2 = (\frac{m}{r^2})^2 (r^2 d\varphi)^2$ follows from that of m/r^2 (which it is very easy to show from the expression for m). Similarly from

$$(e^{2\Psi} - 1)dr^2 = [(e^{2\Psi} - 1)/r^2](rdr)^2$$

one gets analytic contributions since $(e^{2\Psi} - 1)/r^2$ is an analytic function of X and Y for all X,Y.

The determinant of the transformed metric is just $-e^{4\Psi} \neq 0$ so that contravariant components are also everywhere analytic. The velocities in $(XYZt)$ co-ordinates are given by

$$V^X = \frac{Y(y/r)}{(1 - y^2)^{\frac{1}{2}}}$$

$$V^Y = \frac{X(y/r)}{(1 - y^2)^{\frac{1}{2}}}$$

$$V^t = \frac{[1 - m(y/r)]}{(1 - y^2)^{\frac{1}{2}}} \qquad \text{III(4)}$$

Since $(\frac{y}{r})$ is an analytic function of X,Y and $x^2 = \frac{4}{a^2}(X^2 + Y^2)$ and

$$y = \frac{dm}{dr} = \frac{x}{(1 + x^2)^{\frac{1}{2}} + 1} \qquad \left(x = \frac{2}{a} r \right)$$

is less than unity for all finite x and hence all finite X,Y we find from Eqs. I(6) and I(7) that v^u remain analytic for all finite X,Y.

Since $\frac{dm}{dr} < 1$ we have $m < r$ and hence $r^2 - m^2 > 0$ for all r, a fact which will be needed later.

The preceding calculations are given in such meticulous detail because such computations do not appear in most texts, and the results (that m/r^2 and $(e^{2\Psi} - 1)/r^2$ need to be analytic function of r^2) are not obvious without computations. There is no general method for asserting differentiability of a metric except to display it in a co-ordinate system where the components are differentiable and where $(-g)^{\frac{1}{2}} > 0$.

IV. Completeness

In this section we show that our space is complete, i.e. every geodesic has infinite length in both directions; for null geodesics we have to measure the length by means of an affine parameter. The problem is easy essentially because of the high symmetry involved so that we have a large number of constants of motion. If we take as Lagrangian for the general equations

$$L = \tfrac{1}{2}g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} \qquad \text{IV(1)}$$

where λ is a parmeter which can be taken as path length s for non-null lines, we get the following constants of motion

$$E = -P_t = \frac{dt}{d\lambda} - m\frac{d\varphi}{d\lambda}$$

$$P_z = e^{2\Psi}\,dz/d\lambda$$

$$\ell = P_\varphi = (r^2 - m^2)\frac{d\varphi}{d\lambda} + m\frac{dt}{d\lambda}$$

$$\epsilon = -\left(\frac{dt}{d\lambda}\right)^2 + e^{2\Psi}\left\{\left(\frac{dr}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2\right\} + (r^2 - m^2)\left(\frac{d\varphi}{d\lambda}\right)^2$$

$$+ 2m\frac{d\varphi}{d\lambda}\frac{dt}{d\lambda} \qquad \text{IV(2)}$$

where E, ℓ, P_z can be interpreted as energy, angular momentum and momentum along z-direction for a particle of unit mass. We have

$$\epsilon = 0, \pm 1$$

depending on whether the geodesic is null, space-like or time-like.

We can rewrite the above relations as follows

$$\frac{dt}{d\lambda} = E + \frac{m\ell - m^2E}{r^2} = \frac{\ell m + E(r^2 - m^2)}{r^2}$$

$$\frac{d\varphi}{d\lambda} = \frac{\ell - mE}{r^2}$$

$$\frac{dz}{d\lambda} = e^{-2\Psi} \cdot P_z$$

$$\left(\frac{dr}{d\lambda}\right)^2 + e^{-2\Psi}\left[-E^2 - \epsilon + \frac{(\ell - Em)^2}{r^2}\right] + e^{-4\Psi} P_z^{\ 2} \qquad \text{IV(3)}$$

The last equation can be re-written as

$$\tfrac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V_{eff}(r) = 0 \qquad \text{IV(4)}$$

with a suitable definition of $V_{eff}(r)$.

It is to be noted that we must have $\ell = 0$ for a particle passing through origin.

The last of the Eqs. IV(4) is particularly easy to understand as it resembles the motion of a particle in a potential well. If we look at the behaviour of Ψ we see that for large r

$$e^{-\Psi} \sim r^{\frac{1}{4}}.$$

Hence if $P_z \neq 0$, $V_{eff}(r)$ becomes positive for sufficiently large r. Hence motion along r-co-ordinate will be bounded so that there will be a value of r corresponding to all values of λ. Again since motion of r is bounded we find that

$$\frac{dz}{d\lambda} < A$$

where A is a constant depending on the particular geodesic. Hence in z-direction also the particle can escape to infinity only at an infinite value of λ so that with respect to z-co-ordinate the geodesics can be continued for all values of λ. Similar arguments hold for other co-ordinates. It is to be noted that even for particles passing through origin there is no singularity involved as here $\ell = 0$ and m/r^2 is finite at origin.

Next we take up the case $P_z = 0$. In this case as $r \to \infty$

$$V_{eff}(r) \sim -r^{\frac{1}{2}}$$

so that

$$\frac{dr}{d\lambda} \sim r^{\frac{1}{4}}$$

This shows that infinite value of r is reached only when $\lambda \to \infty$. The other equations show that as λ and hence $r \to \infty$, $\frac{dt}{d\lambda}$ and $\frac{d\varphi}{d\lambda} \to 0$ so that with respect to these co-ordinates also λ can be continued to infinite value. Hence we see that in all cases the geodesics can be continued for all values of the path parameter. Hence the space is complete.

V. Absence of Closed Time-Like Lines

If the space contains closed time-like lines then t will be a periodic function of the parameter λ describing the line.

In such a case we must have maxima and minima of t as function of λ so that there will be points where $(\frac{dt}{d\lambda})$ will be zero. At such a point we will have

$$-(\frac{ds}{d\lambda})^2 = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dz^\nu}{d\lambda} = e^{2\Psi}[(\frac{dr}{d\lambda})^2 + (\frac{dz}{d\lambda})^2]$$

$$+ (r^2 - m^2)(\frac{d\varphi}{d\lambda})^2 > 0$$

as both $e^{2\Psi}$ and $(r^2 - m^2)$ are positive for all values of r. Hence at such points the line is no longer time-like. Hence our space does not have closed time-like lines.

VI. Characterization of the Metric

In this section we shall try to characterize our solution by its Killing vectors. In the following, we do not distinguish between vectors X^μ and their corresponding differential operators related by

$$X = X^\mu \frac{\partial}{\partial X^\mu} \qquad \text{VI(1)}$$

A basis for Killing vectors for our space are the following three vectors

$$T = \frac{\partial}{\partial t}$$
$$Z = \partial/\partial z$$
$$\Phi = \frac{\partial}{\partial \varphi} \qquad \text{VI(2)}$$

Each of them satisfies Killing's equations

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \qquad \text{VI(3)}$$

which reduces to

$$\frac{\partial}{\partial x^{\beta}} g_{\mu\nu} = 0 \qquad \text{VI(4)}$$

if ξ is the vector $\partial/\partial x^{\beta}$. If we now try to look at the problem from a more general point of view and try to investigate infinitely long cylindrical systemswhich are stationary, the above are the natural Killing vectors for such a system. Since the Killing vectors commute with each other, we can choose co-ordinate axes $t\varphi z$ so that the Killing vectors point along them. In such a situation the metric components will depend only on the fourth co-ordinate r. If we impose the following reflection symmetries which are appropriate to an infinitely long cylindrically symmetric system, which is rotating, we can, with one further restriction on g_{tt}, arrive at our form of line-element.

We impose two reflection symmetries. The first is $z \rightarrow -z$. The second is the simultaneous reflection $t \rightarrow -t$ and

$\varphi \to -\varphi$. . The first one eliminates all cross terms in z. For consider the term g_{rz} and make the transformation $z' = -z$, other co-ordinates remaining same. This gives

$$g_{rz'} = -g_{rz}$$

But from the reflection symmetry

$$g_{rz'} = g_{rz}$$

Hence $g_{rz} = 0$. Similarly using the other symmetry all cross terms except the one in $\varphi - t$ are eliminated. Hence our metric takes the form

$$g_{tt}dt^2 + g_{rr}dr^2 + g_{zz}dz^2 + g_{\varphi\varphi}d\varphi^2 + 2g_{\varphi t}d\varphi dt \qquad \text{VI(5)}$$

where each of the components depends only on r. Now by a simple scale transformation for r, we can make $g_{rr} = g_{zz}$. This does not change any of the symmetries and hence leaves the above form unchanged. Next we make the simplifying assumption that $g_{tt} = -1$.

Hence calling $g_{rr} = g_{zz} = e^{2\Psi}$, $g_{t\varphi} = m$, we have

$$-ds^2 = e^{2\Psi}(dr^2 + dz^2) + g_{\varphi\varphi}d\varphi^2 + 2md\varphi dt - dt^2 \qquad \text{VI(6)}$$

Now the field equations give

$$g_{\varphi\varphi} = r^2 - m^2$$

Hence we get

$$-ds^2 = e^{2\Psi}(dr^2 + dz^2) + (r^2 - m^2)d\varphi^2 + 2md\varphi dt - dt^2 \qquad \text{VI(7)}$$

Even with this form of line element, there are two solutions of the field equations. If we use co-moving frame we get van Stockum's solution which represent matter in rigid rotation. If we employ our form of energy-momentum tensor we get solution for matter in non-rigid rotation invariantly distinguished from van Stockum's solution by the presence of shear.

An invariant way of restating our special condition $g_{tt} = -1$ is to demand that the time-like Killing vector T has a constant magnitude.

$$T \cdot T = -1 \qquad \text{VI(8)}$$

A constant value of $T.T$ implies that that congruence of curves to which the time-like Killing vector is tangent is a geodesic congruence.

It may be remarked that if we relax the condition that g_{tt} be a constant, we will get a family of solutions, getting in general two solutions for a given choice of the function $g_{tt}(r)$.

We sum up the contents of this section by giving below the conditions that uniquely lead to our form of the metric.

a) There exist three commuting Killing vectors T, Φ, Z.

b) Our system has reflection symmetries appropriate to a rotating cylinder of infinite length - i.e. it is invariant under the following conditions: $(t, r, \varphi, z) \to (t, r, \varphi, -z)$, $(t, r, \varphi, z) \to (-t, r, -\varphi, z)$.

c) $g_{tt} = -1$.

d) The system is in non-rigid rotation, i.e. shear is present.

VII. Co-moving Co-ordinates and Cosmology

One can transform the line-element I(1) to a form in which the velocity-vector is $V^{\mu} = \delta^{\mu}{}_{o}$ (co-moving frame). In this system, the metric explicitly involves time and is no longer stationary. It has the following form in the co-moving system

$$-ds^2 = -dt^2 + \{e^{2\Psi} + \frac{t^2}{a^2} \cdot \frac{1}{1 + x^2} y^4\}dr^2 + e^{2\Psi}dz^2 + (r^2 - m^2)d\varphi^2$$

VII(1)

$$+ \frac{2}{(1 + x^2)^{\frac{1}{2}}} (\frac{y}{a})^{3/2}(r - my)r^{\frac{1}{2}}tdrd\varphi + 2(r/ay)^{\frac{1}{2}}(ry - m)d\varphi dt$$

Although the expansion vanishes the non-vanishing shear would in this solution give rise to a Doppler shift in the frequency of light emitted by a particle and received by another - put in another way the non-stationary nature of the metric in the co-moving system would cause a spectral shift. However this Doppler shift would in general be strongly anisotropic unlike the actually observed more or less isotropic Hubble red-shift. We do not therefore propose the solution as a model of the observed universe but as noted earlier we can hope to build singularity-free dynamical model from this. This solution further emphasizes that one can construct anti-Mach metrics without taking recourse to the Λ-term or introducing unphysical situations.

ACKNOWLEDGEMENTS

I wish to express my deep gratitude to Professor A.K. Raychaudhuri for suggesting the problem and for numerous helpful discussions. I also wish to thank Professor C.W. Misner for many helpful suggestions, and Professor J. Ehlers for reading part of the manuscript and suggesting improvements. The research was supported by NASA Grant NsG 436.